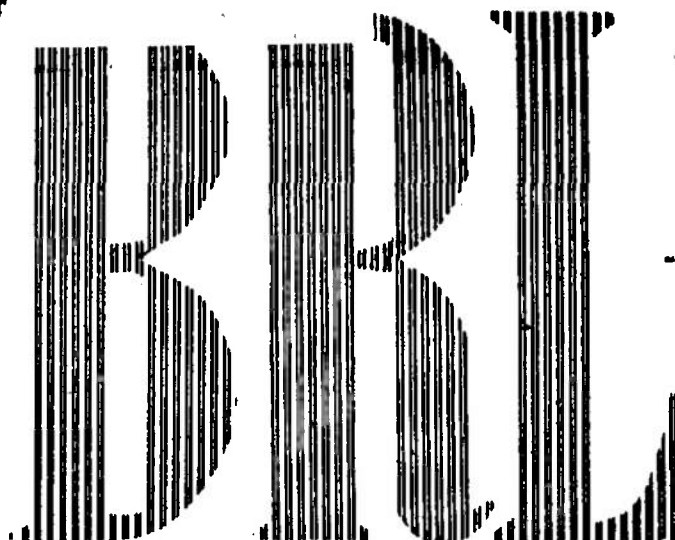


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APPLICATION OF A MARKOV CHAIN TO A GENERALIZATION OF
AN ALGEBRAIC IDENTITY

C. Masaitis

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APPLICATION OF A MARKOV CHAIN TO A GENERALIZATION OF AN ALGEBRAIC IDENTITY

ABSTRACT

An identity
$$\frac{x_1^n - x_2^n}{x_1 - x_2} = \sum_{i_1+i_2=n-1} x_1^{i_1} x_2^{i_2}$$

is here generalized to an arbitrary number of variables

x_1, \dots, x_m . The proof of the generalized identity

$$\sum_{i_1+i_2+\dots+i_m=n-m+1} x_1^{i_1} x_2^{i_2} \dots x_m^{i_m} = \frac{x_1^n}{(x_1-x_2)(x_1-x_3)\dots(x_1-x_m)} +$$

$$\frac{x_2^n}{(x_2-x_1)(x_2-x_3)\dots(x_2-x_m)} + \dots + \frac{x_m^n}{(x_m-x_1)(x_m-x_2)\dots(x_m-x_{m-1})}$$

is obtained by computing transition probabilities in a Markov Chain.

Assume that $0 < x_i < 1$, $i = 1, 2, \dots, m$, $x_i \neq x_j$ for $i \neq j$, and let $y_i = 1 - x_i$. Consider a Markov Chain with the matrix of transition probabilities:

$$\begin{pmatrix} x_1 & y_1 & 0 & 0 & 0 & \dots & 0 \\ 0 & x_2 & y_2 & 0 & 0 & \dots & 0 \\ 0 & 0 & x_3 & y_3 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & x_m \end{pmatrix}$$

If $n \geq r$ then a transition of the system from the state k to a state $k + r$ in n steps can occur in any of the set of mutually exclusive ways, each corresponding to a distinct sequence of integers t_1, t_2, \dots, t_r , satisfying the inequalities $1 \leq t_1 < t_2 < \dots < t_r \leq n$. In each of these cases the system passes from the state $k + h$ to the state $k + h + 1$ at the step number t_{h-1} . The probability of such an event is $y_k y_{k+1} \dots y_{k+r-1} x_k^{t_1-1} x_{k+1}^{t_2-t_1-1} \dots x_{k+r}^{n-t_r}$. Here the sum of the exponents of x_{k+i} 's is $t_1 - 1 + t_2 - t_1 - 1 + \dots + n - t_r = n - r$. Therefore the probability $p_{kr}^{(n)}$ of transition in n steps from the state k to the state $k + r$ is given by

$$(1) \quad p_{kr}^{(n)} = y_k y_{k+1} \dots y_{k+r-1} \sum_{i_1 + \dots + i_{r+1} = n-r} x_k^{i_1} x_{k+1}^{i_2} \dots x_{k+r}^{i_{r+1}}$$

On the other hand, $p_{kr}^{(n)}$ can be obtained by the method described in

[1]. Let $D(s)$ be the determinant of the system

$$\begin{aligned} \xi_j &= s(x_j \xi_j + y_j \xi_{j+1}), \quad j = 1, 2, \dots, m-1, \\ (2) \quad \xi_m &= s x_m \xi_m, \end{aligned}$$

and let $(\xi_1^{(t)}, \dots, \xi_m^{(t)})$ be a non-trivial solution of (2) corresponding to the root s_t of $D(s)$. Similarly, let $(\eta^{(t)}, \dots, \eta_m^{(t)})$ be a non-trivial solution of

$$\begin{aligned} (3) \quad \eta_1 &= s x_1 \eta_1, \\ \eta_i &= s(y_{i-1} \eta_{i-1} + x_i \eta_i), \end{aligned}$$

corresponding to the same root s_t of $D(s)$.

Let

$$(4) \quad c_t = \frac{1}{\sum_{v=1}^m \xi_v^{(t)} \eta_v^{(t)}},$$

and

$$(5) \quad \zeta_{ji}^{(t)} = c_t \xi_j^{(t)} \eta_i^{(t)}.$$

Then

$$(6) \quad p_{kr}^{(n)} = \frac{\zeta_{k \ k+r}^{(1)}}{s_1^n} + \frac{\zeta_{k \ k+r}^{(2)}}{s_2^n} + \dots + \frac{\zeta_{k \ k+r}^{(m)}}{s_m^n}.$$

It follows from (2) that non-trivial solutions $(\xi_1^{(t)}, \dots, \xi_m^{(t)})$ can be obtained for the following values of s : $s_1 = \frac{1}{x_m}$, $s_2 = \frac{1}{x_{m-1}}$, \dots , $s_m = \frac{1}{x_1}$. By substituting $s_t = \frac{1}{x_{m-t+1}}$ in (2) the following expressions are obtained:

$$(7) \quad \xi_m^{(t)} = \xi_{m-1}^{(t)} = \dots = \xi_{m-t+2}^{(t)} = 0 ,$$

$$(8) \quad \xi_{m-t+1}^{(t)} = 1 ,$$

The substitution of (8) in the first equation of (2) yields

$$(9) \quad \xi_{m-t} = \frac{y_{m-t}}{x_{m-t+1} - x_{m-t}}$$

Suppose, by induction, that for $j \leq m - t$

$$(10) \quad \xi_j^{(t)} = \prod_{p=0}^{m-t-j} \frac{y_{j+p}}{x_{m-t+1} - x_{j+p}}$$

The substitution of (10) in (2) yields

$$\xi_{j-1}^{(t)} = \prod_{p=0}^{m-t-j+1} \frac{y_{j+p}}{x_{m-t+1} - x_{j+p}}$$

Thus, in view of (9), (10) holds for every $j \leq m - t$.

Similarly, it follows from (3) that

$$(11) \quad \begin{aligned} \eta_i^{(t)} &= 0 , \quad \text{for } i < m - t + 1 ; \\ \eta_{m-t+1}^{(t)} &= 1 ; \\ \eta_i^{(t)} &= \prod_{p=0}^{i-m+t-2} \frac{y_{i-p-1}}{x_{m-t+1} - x_{i-p}} , \quad \text{for } i > m - t + 1 \end{aligned}$$

Therefore it follows from (4), (7), (8), and (11) that

$$(12) \quad c_t = 1 , \quad t = 1, 2, \dots, m.$$

Now (5), (7), (8), (10), and (12) yield

$$\begin{aligned}
 \zeta_{ji}^{(t)} &= 0, \text{ if } i < m - t + 1 \text{ or } j > m - t + 1; \\
 \zeta_{m-t+1, m-t+1}^{(t)} &= 1; \\
 \zeta_{m-t+1, i}^{(t)} &= \prod_{p=0}^{m-t-i-2} \frac{y_{i+p-1}}{x_{m-t+1} - x_{i+p}}, \text{ if } i > m - t + 1; \\
 \zeta_{j, m-t+1}^{(t)} &= \prod_{p=0}^{m-t-j} \frac{y_{j+p}}{x_{m-t+1} - x_{j+p}}, \text{ if } j < m - t + 1; \\
 \zeta_{ij}^{(t)} &= \prod_{p=0}^{m-t-j} \frac{y_{j+p}}{x_{m-t+1} - x_{j+p}} \prod_{q=0}^{i-m+t-2} \frac{y_{i-q-1}}{x_{m-t+1} - x_{j-q}}, \\
 &\text{if } i > m - t + 1 \text{ and } j < m - t + 1.
 \end{aligned}
 \tag{12}$$

Therefore by (6)

$$\begin{aligned}
 P_{1,m}^{(n)} &= y_1 y_2 \dots y_{m-1} \left[\frac{x_m^n}{(x_m - x_{m-1})(x_m - x_{m-2}) \dots (x_m - x_1)} + \right. \\
 &+ \frac{x_{m-1}^n}{(x_{m-1} - x_m)(x_{m-1} - x_{m-2}) \dots (x_{m-1} - x_1)} + \dots + \\
 &\left. + \frac{x_1^m}{(x_1 - x_m)(x_1 - x_{m-1}) \dots (x_1 - x_2)} \right]
 \end{aligned}
 \tag{13}$$

$$\text{Let } P_k(x_1, x_2, \dots, x_m) = \sum_{i_1 + i_2 + \dots + i_m = k} x_1^{i_1} x_2^{i_2} \dots x_m^{i_m}$$

Then by (1) and (13)

$$(14) \quad P_{n-m+1}(x_1, x_2, \dots, x_m) = \frac{\begin{vmatrix} 1 & 1 & \dots & 1 \\ x_1 & x_2 & \dots & x_m \\ x_1^2 & x_2^2 & \dots & x_m^2 \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{m-2} & x_2^{m-2} & \dots & x_m^{m-2} \\ x_1^n & x_2^n & \dots & x_m^n \end{vmatrix}}{V(x_1, x_2, \dots, x_m)},$$

where V is a Vandermonde's determinant. Now the numerator N of the right hand side member of (14) is a polynomial in x_1, \dots, x_m and it vanishes whenever $x_i = x_j$ for $i \neq j$. Since $V = (x_1 - x_2) \dots (x_1 - x_m)(x_2 - x_3) \dots (x_{m-1} - x_m)$, it follows that N is divisible by V . Hence (14) is an equality of two polynomials which is valid for every x_i such that $0 < x_i < 1$. Hence (14) is an identity whenever N/V is defined, i.e. whenever $x_i \neq x_j$ for $i \neq j$. This identity can be written in the form

$$(15) \quad P_{n-m+1} = \frac{x_1^n}{(x_1 - x_2)(x_1 - x_3) \dots (x_1 - x_m)} + \frac{x_2^n}{(x_2 - x_1)(x_2 - x_3) \dots (x_2 - x_m)} + \dots + \frac{x_m^n}{(x_m - x_1)(x_m - x_2) \dots (x_m - x_{m-1})}.$$

For $m = 2$ (15) reduces to

$$x_1^{n-1} + x_1^{n-2}x_2 + x_1^{n-3}x_2^2 + \dots + x_2^{n-1} = \frac{x_1^n}{x_1 - x_2} + \frac{x_2^n}{x_2 - x_1}.$$

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